Fourier Analysis Feb 08, 2022

1. Review.

Recall that a good kernel means a sequence $(K_n)_{n=1}^{\infty}$ of integrable functions on the circle satisfying

① $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$;

② $\exists\ M > 0$ such that $\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$;

③ For any $0 < \delta < \pi$,

$$\int_{\delta < |x| < \pi} |K_n(x)|\,dx \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

(Convergence Thm for good kernels):

Let $(K_n)_{n=1}^{\infty}$ be a good kernel on the circle. Suppose f is integrable on the circle. Then

$$K_n * f(x) \longrightarrow f(x) \quad \text{if } f \text{ is cts at } x.$$

If f is cts on the circle, then

$$K_n * f(x) \rightrightarrows f(x) \quad \text{on the circle.}$$

Example 1. (Fejer's Kernel)

For $N \in \mathbb{N}$, set

$$F_N(x) = \frac{\sin^2(\frac{N}{2}x)}{N \sin^2(\frac{x}{2})}$$

$$= \frac{D_0(x) + \cdots + D_{N-1}(x)}{N} \quad \left(\text{Recall } D_n(x) = \sum_{k=-n}^{n} e^{ikx} \right)$$

$$= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{inx}$$

We call F_N the N-th Fejer's kernel.

Corollary (Fejer's thm)

Let f be integrable on the circle, then

- $F_N * f(x) \longrightarrow f(x)$ if f is cts
- If f is cts on the circle, then

$$F_N * f \rightrightarrows f \quad \text{on the circle.}$$

Recall that

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{inx}$$

So

$$F_N * f(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \cdot \hat{f}(n)\, e^{inx}$$

We also write

$$\sigma_N f(x) := F_N * f(x)$$

(we call it the N-th Cesáro mean of the Fourier series of f).

Corollary 1 : Let f be cts on $[-\pi, \pi]$ with $f(\pi) = f(-\pi)$.

Then $\forall\, \varepsilon > 0$, $\exists$ a trigonometric poly $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Pf. $\left.\begin{array}{l} \text{Since } f \text{ is cts on the circle,} \\ \qquad \underbrace{\sigma_N f}_{\text{A triogometric poly}} \rightrightarrows f \quad \text{on the circle.} \end{array}\right\}$ by Fejér's Thm.

(uniqueness Thm for Fourier Series)

Corollary 2. Suppose f is cts on the circle such that $\hat{f}(n) \equiv 0$ for $n \in \mathbb{Z}$. Then $f \equiv 0$.

Pf. Since $\hat{f}(n) \equiv 0$ for $n \in \mathbb{Z}$,

$$\sigma_N f(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{inx}$$

$$= 0 \qquad \text{for all } N \in \mathbb{N}.$$

But by Fejer's Thm

$$\sigma_N f \rightrightarrows f \quad \text{as } N \to \infty$$

which implies $f \equiv 0$.

- Similarly, we can define "good kernel" for a family of integrable functions

$$(K_t)_{t \in (a,b)}$$

as $t \to t_0$. More precisely, $(K_t)_{t \in (a,b)}$ is said to be a good kernel on the circle as $t \to t_0$ if

① $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_t(x)\,dx = 1$ for all $t \in (a,b)$

② $\exists\, M>0$, such that

$$\int_{-\pi}^{\pi} |K_t(x)|\,dx \leq M \quad \text{for all } t \in (a,b)$$

③ For $0<\delta<\pi$,

$$\lim_{t\to t_0} \int_{\delta<|x|<\pi} |K_t(x)|\,dx = 0.$$

(Convergence Thm): If $(K_t)_{t\in(a,b)}$ is a good kernel as $t \to t_0$, then

$$K_t * f(x) \xrightarrow{(t\to t_0)} f(x) \quad \text{if } f \text{ is cts at } x_0$$

- If f is cts on the circle

$$K_t * f \rightrightarrows f \quad \text{as } t \to t_0.$$

Example 2. Define for $t \in (0, \pi)$,

$$\varphi_t(x) = \begin{cases} \frac{\pi}{t} & \text{if } 0 \leq |x| < t \\ 0 & \text{if } |x| \geq t. \end{cases}$$

Then $(\varphi_t)_{t \in (0,\pi)}$ is a good kernel as $t \to 0$.

Check :

- $\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_t(x)\, dx$

$$= \frac{1}{2\pi} \int_{-t}^{t} \frac{\pi}{t}\, dx = 1$$

- $\varphi_t(x) \geq 0$

- $\int_{\delta < |x| \leq \pi} \varphi_t(x)\, dx$

$$= \int_{\{x : \delta < |x| < \pi\} \cap (-t, t)} \frac{\pi}{t}\, dx$$

$= 0$ when t is small enough.

Check:

$$f * \varphi_t(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_t(y)\, f(x-y)\, dy$$

$$= \frac{1}{2\pi}\int_{-t}^{t} \frac{\pi}{t} \cdot f(x-y)\, dy$$

$$= \frac{1}{2t}\int_{-t}^{t} f(x-y)\, dy$$

Example 3 (Poisson Kernel on the circle).

For $r \in (0,1)$, define

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

We call $(P_r)_{r\in(0,1)}$ the Poisson kernel on the circle as $r \to 1$.

Lem 3. For $r \in (0,1)$

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}.$$

Pf.

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}$$

$$= \left(\sum_{n=-\infty}^{-1} r^{|n|} e^{inx}\right) + 1 + \left(\sum_{n=1}^{\infty} r^{n} e^{inx}\right)$$

$$= 1 + \sum_{n=1}^{\infty} r^{+n} e^{-inx} + \sum_{n=1}^{\infty} r^n e^{inx}$$

$$= 1 + \sum_{n=1}^{\infty} (re^{-ix})^n + \sum_{n=1}^{\infty} (re^{ix})^n$$

$$= 1 + \frac{re^{-ix}}{1-re^{-ix}} + \frac{re^{ix}}{1-re^{ix}}$$

$\left(\text{using } \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \text{ for } |z|<1\right)$

$$= 1 + \frac{re^{-ix}(1-re^{ix}) + re^{ix}(1-re^{-ix})}{(1-re^{-ix})(1-re^{ix})}$$

$$= 1 + \frac{re^{-ix} - r^2 + re^{ix} - r^2}{1 - re^{-ix} - re^{ix} + r^2}$$

$$= 1 + \frac{2r\cos x - 2r^2}{1 - 2r\cos x + r^2}$$

(using $e^{-ix} + e^{ix} = 2\cos x$)

$$= \frac{1 - r^2}{1 - 2r\cos x + r^2}. \qquad \blacksquare$$

$$1 - 2r\cos x + r^2 = (1 - r\cos x)^2 + r^2(1 - \cos^2 x) > 0$$

Check: (Poisson kernel is good)

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)\,dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} \, dx \qquad (*)$$

Since the series converges unif. on ~~the~~ circle, we have

$$(*) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} r^{|n|} e^{inx} \, dx$$

$$= 1$$

- $$\int_{-\pi}^{\pi} |P_r(x)| \, dx$$

$$= \int_{-\pi}^{\pi} P_r(x) \, dx = 2\pi.$$

- Let $0 < \delta < \pi$.

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$$

$$= \frac{1-r^2}{(1-r)^2 + 2r(1-\cos x)}$$

If $\delta < |x| < \pi$, then $1-\cos x \geq 1-\cos\delta > 0$

So $$P_r(x) \leq \frac{1-r^2}{2r(1-\cos x)}$$

$$\leq \frac{1-r^2}{2r(1-\cos\delta)}$$

Hence

$$\int_{\delta<|x|<\pi} P_r(x)\,dx \leq \frac{1-r^2}{2r(1-\cos\delta)} \cdot 2\pi$$

$$\to 0 \quad \text{as} \quad r \to 1 .$$

Hence $(P_r)_{r\in(0,1)}$ is a good kernel as $r\to 1$.

Now as a direct consequence of the convergence Thm for good kernels, we have

Corollary 4: Let f be integrable on the circle.

Then

(1) $$P_r * f(x) \to f(x) \quad \text{as } r\to 1$$

whenever f is cts at x;

(2) If f is cts on the circle, then

$$P_r * f(x) \rightrightarrows f(x) \quad \text{on the circle}$$

as $r \to 1$.

Lem 5. Let f be integrable on the circle.

Then
$$P_r * f(x) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{inx},$$
$r \in (0, 1)$.

Pf. Given $0 < r < 1$.

$$\sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \cdot e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-in(x-y)} dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(y) r^{|n|} e^{-in(x-y)} dy$$

(Reason: $\sum_{n=-\infty}^{\infty} f(y) r^{|n|} e^{-in(x-y)}$ converges unif. on the circle)

$$\left(P_r(y-x) = P_r(x-y) \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cdot P_r(x-y)\, dy$$

$$= P_r * f(x). \qquad \blacksquare$$

We rewrite $A_r f(x) := P_r * f(x)$

and call it the Abel mean of the Fourier series of f.